1. Motivation

2. Measurements and Modeling of emittance evolution of Fermilab Booster

3. Implications on the non-scaling FFAG accelerators

4. Possible measurements of scaling laws at UMER


Motivation

- What are mechanisms of emittance growth in synchrotrons? Envelope instability or half integer stopband; Montague resonances; linear coupling; nonlinear resonances; integer resonances; ….

- Measurements? Can one measure emittance turn-by-turn?

- FFAG → a favorable candidate for high power proton drivers: Constant guide field → high rep rate, in the kHz.
  - Scaling design
  - Non-scaling design.
    - Beam quality can deteriorate when crossing resonances.
    - Tune-ramp rate: $\sim -10^{-3}$ to $-10^{-2}$ per turn

\[
\frac{\Delta v_{x,z}}{\Delta n} \approx -\frac{v_x}{\beta^2 E} \frac{\Delta E}{\Delta n}
\]
Ionization beam profile monitors (IPM's) measure the beam profile by collecting electrons from background gas ionization. Measurement accuracy will depend on the number of electrons collected. The number of electrons collected depends on pressure, bunch intensity, and data collection mode. The center-to-center distance between microchannels is of the order of 1.5mm. Thus one can measure the beam width to within 1.5-2.0 mm. For the IPM at Fermilab Booster, the space charge correction has been shown to be important [J. Amundsen et al PRSTAB 6 102801 (2003)]

\[ \varepsilon = \frac{\sigma^2}{\beta} \]
Typical Injection scheme at the Fermilab Booster: H- at 400 MeV from Linac is about 30 mA. Strip injection into the Booster gives about $4.2 \times 10^{11}$ protons per injection turn. At 400 MeV, the revolution frequency is $4.51 \times 10^5$ Hz. The Normal operation has 12 injection turns, and the total intensity is about $5 \times 10^{12}$ ppp at 15 Hz rep-rate. In fact, when the injection turn is larger than 12, the Booster loss problem becomes severe.
Space charge effects: Linac delivers about 30 mA beam current to the Fermilab Booster, i.e. about $4.2 \times 10^{11}$ particles in one injection turn.

\[ \varepsilon = a_0 + b_1 t + b_2 \int_0^t K_{sc} \, dt \]

\[ K_{sc} = \frac{2Nr_0}{\beta^2 \gamma^3} \]
Horizontal IPM measurements
\[ \sigma^2_x = \beta_x \varepsilon_{\text{rms}} + D^2 \delta^2 \]
\[ = a + bt + ct^2 + A e^{-\alpha t} \cos[2\pi(f_1 t + f_2 t^2) + \phi] \]
\[ a + bt + ct^2 = \beta_x \varepsilon_{\text{rms}} + D^2 \bar{\delta}^2 \]
\[ f_{\text{syn}}(t) = f_1 + 2f_2 t \]
\[ 2A = D^2 (\delta_1^2 - \delta_2^2) \]
\[ \delta_1 = \frac{\gamma_T}{3^{1/6} \beta \tau_{\text{ad}} \Gamma(2/3) \frac{2\Lambda_{t\Delta E}}{3mc^2 \gamma}} \]
\[ \delta_2 = \frac{\nu_s A \phi \delta}{\hbar \eta \pi \delta_1} \]
\[ A_{\phi \delta} = \frac{h \omega_0}{\beta^2 \gamma_T mc^2} A_{t\Delta E} \]
\[ \tau_{\text{ad}} = \left( \frac{\pi \beta^2 \gamma_T^4 mc^2}{\gamma \omega_0^2 heV \cos \phi_s} \right)^{1/3} \]
\[ \tau_{\text{ad}} = \left( \frac{\pi \beta^2 mc^2 \gamma_T^4}{\gamma \omega_0^2 heV \cos \phi_s} \right)^{1/3} \approx 0.20 \text{ ms}, \]
\[ \tau_{\text{nl}} = \gamma_T^3 \eta_1 \hat{\delta} = \gamma_T \frac{3 \beta_0^2 + \gamma_T^2 \alpha_1}{2\gamma} \hat{\delta} \approx 0.07 \text{ ms}, \]
\[ \delta = (\Delta p/p)_{\text{rms}} \]
Summary:
1. \( \frac{d\varepsilon_z}{dt} \sim b + K_{sc} \)
2. The horizontal emittance and the off-momentum spread can be separated by using different scaling law (energy dependence).
3. The **horizontal emittance is less affected by the space charge force!** Why?
4. The slow linear growth of the horizontal emittance is the same as that of the vertical plane! \( \varepsilon_z \) increases linearly with \( t \) at about 1 \( \pi \)-mm-mrad in \( 10^4 \) revolutions.

5. The post-transition horizontal bunch-width oscillation is induced essentially by the longitudinal mis-match of the bunch shape with space charge effect in the rf potential well. Using the bunch shape mis-match, one can deduce the phase space area.
An Overview of Booster’s lattice

- Superperiod = 24
- In each period, four Combined-function magnets. Length 2.889 m. Gradients are $K_1(F) = 0.0542 \text{ m}^{-2}$ and $K_1(D) = -0.0577 \text{ m}^{-2}$. The integrated focusing strengths are $0.313 \text{ m}^{-1}$ and $-0.333 \text{ m}^{-1}$.
- Horizontal tune 6.7, vertical tune 6.8
- Extraction doglegs in L03 and L13 perturb the lattice functions

\[
\beta_{x,\text{max}} = 33.7 \text{ m} \quad \beta_{z,\text{max}} = 20.4 \text{ m} \quad D_{\text{max}} = 3.2 \text{ m}
\]
## The Fermilab Booster

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference (m)</td>
<td>474.2</td>
</tr>
<tr>
<td>Inj/ext energy (GeV)</td>
<td>0.4/8.0</td>
</tr>
<tr>
<td>Cycling rate (Hz)</td>
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<tr>
<td>Hori/vert betatron tunes</td>
<td>6.7/6.8</td>
</tr>
<tr>
<td>Superperiod</td>
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<tr>
<td>Transition gamma</td>
<td>5.446</td>
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<tr>
<td>Harmonic number</td>
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</tr>
<tr>
<td>Protons per pulse</td>
<td>5.0E12</td>
</tr>
<tr>
<td>Longitudinal 95% emittance (eV-s)</td>
<td>0.10 at injection</td>
</tr>
<tr>
<td>Transverse 95% emittance (mm-mr)</td>
<td>12π (normalized) at injection</td>
</tr>
</tbody>
</table>
Space charge Modeling algorithm:

We consider N-particle in Gaussian distribution and construct a model with 24 Superperiod FODO cells.

\[
M_{D\rightarrow F} = \begin{pmatrix}
\sqrt{\frac{\beta_{z,F}}{\beta_{z,D}}} \cos \psi_x & \sqrt{\beta_{x,F} \beta_{x,D}} \sin \psi_x & 0 & 0 \\
-\frac{1}{\sqrt{\beta_{x,F} \beta_{x,D}}} \sin \psi_x & \sqrt{\frac{\beta_{z,D}}{\beta_{z,F}}} \cos \psi_x & 0 & 0 \\
0 & 0 & \sqrt{\frac{\beta_{z,F}}{\beta_{z,D}}} \cos \psi_z & \sqrt{\beta_{x,F} \beta_{x,D}} \sin \psi_x \\
0 & 0 & -\frac{1}{\sqrt{\beta_{x,F} \beta_{x,D}}} \sin \psi_z & \sqrt{\frac{\beta_{z,D}}{\beta_{z,F}}} \cos \psi_z
\end{pmatrix}
\]

\[
M_{F\rightarrow D} = \begin{pmatrix}
\sqrt{\frac{\beta_{z,D}}{\beta_{z,F}}} \cos \psi_x & \sqrt{\beta_{x,F} \beta_{x,D}} \sin \psi_x & 0 & 0 \\
-\frac{1}{\sqrt{\beta_{x,F} \beta_{x,D}}} \sin \psi_x & \sqrt{\frac{\beta_{z,F}}{\beta_{z,D}}} \cos \psi_x & 0 & 0 \\
0 & 0 & \sqrt{\frac{\beta_{z,D}}{\beta_{z,F}}} \cos \psi_z & \sqrt{\beta_{x,F} \beta_{x,D}} \sin \psi_x \\
0 & 0 & -\frac{1}{\sqrt{\beta_{x,F} \beta_{x,D}}} \sin \psi_z & \sqrt{\frac{\beta_{z,F}}{\beta_{z,D}}} \cos \psi_z
\end{pmatrix}
\]
Obtained by ICA method
Space charge force is a local kick on every half cell (no change even when more space charge kicks are added):

\[ \rho(x, z) = \frac{Ne}{2\pi\sigma_x\sigma_z} \exp\left\{ -\frac{x^2}{2\sigma_x^2} - \frac{z^2}{2\sigma_z^2} \right\}, \]

\[ V(x, z) = \frac{Nr_0}{\beta^2\gamma^3} \int_0^\infty \frac{1 - \exp\left\{ -\frac{x^2}{2\sigma_x^2+t} - \frac{z^2}{2\sigma_z^2+t} \right\}}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}} dt \]

\[ \approx \frac{Nr_0}{\beta^2\gamma^3} \left( \frac{x^2}{\sigma_x(\sigma_x + \sigma_z)} + \frac{z^2}{\sigma_z(\sigma_x + \sigma_z)} \right) - \frac{Nr_0}{4\beta^2\gamma^3\sigma_x^2(\sigma_x + \sigma_z)^2} \left( \frac{2 + R}{3} x^4 + \frac{2}{R} x^2 z^2 + \frac{1 + 2R}{3R^3} z^4 \right) + \ldots \]
Space Charge Force

\[
\frac{\Delta x'}{\Delta s} - j \frac{\Delta z'}{\Delta s} = \frac{K_{sc}}{2} \hat{F}_{sc},
\]

\[
\hat{F}_{sc} = j \frac{\sqrt{2\pi}}{\sqrt{\left(\sigma_x^2 - \sigma_z^2\right)}} \left[ w(a + jb) - e^{-(a+jb)^2+(ar+jb)^2} w(ar + \frac{b}{r}) \right],
\]

Where,

\[
w(Z) = e^{-Z^2} \left[ 1 + \frac{2j}{\sqrt{\pi}} \int_0^Z e^{\xi^2} d\xi \right].
\]

But, for the round beam

\[
F_{x,sc} = -\frac{\partial V_{sc}}{\partial x} = K_{sc} \frac{x}{\sigma_x^2} f_x(x^2, z^2)
\]

\[
F_{z,sc} = -\frac{\partial V_{sc}}{\partial z} = K_{sc} \frac{z}{\sigma_z^2} f_z(x^2, z^2)
\]

\[
f_x(x^2, z^2) = \frac{1}{1-r^2} \int_r^1 dv e^{-a^2(1-v^2)-b^2(\frac{1}{v^2}-1)}
\]

, where \( a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, b = \frac{z}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, v^2 = \frac{2\sigma_z^2 + t}{2\sigma_x^2 + t} \)

\[
f_z(x^2, z^2) = \frac{r^2}{1-r^2} \int_r^1 \frac{dv}{v^2} e^{-a^2(1-v^2)-b^2(\frac{1}{v^2}-1)}
\]

\[\text{singularity when } r=1\]
Define \( \varepsilon = 1 - r = \frac{\sigma_x - \sigma_z}{\sigma_x} \), then

\[
f_x(x^2, z^2) = \frac{\sigma_x}{\sigma_x + \sigma_z} \int_0^1 dt \exp\left[-\frac{x^2 t(1 - \varepsilon t / 2)}{\sigma_x (\sigma_x + \sigma_z)} - \frac{z^2 t(1 - \varepsilon t / 2)}{\sigma_x (\sigma_x + \sigma_z)(1 - \varepsilon t)^2}\right]
\]

Do Taylor's expansion and use the integral:

\[
f_n(w) = \int_0^1 e^{-wt} t^n dt = \frac{n!}{w^{n+1}} (1 - e^{-w} \sum_{k=0}^{n} \frac{w^k}{k!}) 2), \text{ where}
\]

\[
w = p + q, \quad p = \frac{x^2}{\sigma_x (\sigma_x + \sigma_z)}, \quad q = \frac{z^2}{\sigma_x (\sigma_x + \sigma_z)}
\]

We obtain

\[
f_x(x^2, z^2) = \frac{\sigma_x}{\sigma_x + \sigma_z} \{ f_0(w) + \varepsilon \frac{p - 3q}{2} f_2(w) + \varepsilon^2 [-2pq f_3(w) + \frac{(p - 3q)^2}{8} f_4(w)]
\]

\[
+ \varepsilon^3 \left[-\frac{5q}{2} f_4(w) - (p - 3q)q f_5(w) + \frac{(p - 3q)^3}{48} f_6(w)\right]
\]

\[
+ \varepsilon^4 \left[-3pq f_5(w) - \frac{5pq - 23q^2}{4} f_6(w) - \frac{(p - 3q)^2 q}{4} f_7(w) - \frac{(p - 3q)^4}{384} f_8(w)\right]
\]

\[
+ \varepsilon^5 \left[-\frac{7}{2} pq f_6(w) - \frac{3pq - 19q^2}{2} f_7(w) - \frac{5p^2 q - 46pq^2 + 93q^3}{16} f_8(w)\right]
\]

\[\left.-\frac{(p - 3q)^3 q}{24} f_9(w) + \frac{(p - 3q)^5}{3840} f_{10}(w) + O(\varepsilon^6)\right\}.
\]
Sextupole nonlinearity on each half cell for nonlinearity in dipoles
- Linear coupling,
- Random quadrupoles with zero tune shifts
- Random closed orbit error
- Dynamical aperture of 80 by 50 pi-mm-mrad

\[ x'' + K_x(s)x = \frac{b_0(s)}{\rho} + \frac{b_1(s)}{\rho} x + \frac{a_1(s)}{\rho} z + \frac{1}{2} \frac{b_2(s)}{\rho} (x^2 - z^2) \]

\[ z'' + K_z(s)z = -\frac{a_0(s)}{\rho} - \frac{b_1(s)}{\rho} z + \frac{a_1(s)}{\rho} x - \frac{b_2(s)}{\rho} xz \]

Random number generators are used to generate \( b_0, a_0, b_1, \) and \( a_1. \) The quadrupole error is subject to a constraint with zero tune shift. The integrated sextupole strengths are set to the systematic values: -0.0173 m\(^{-2}\) and -0.263 m\(^{-2}\) for focusing and defocusing dipoles respectively.

\[ \delta_{\text{rms}}(n) = \delta_{\text{rms}}(1) B_f(n) (1 + (G_\delta - 1)(1 - \exp(-\alpha_g(n - n_t)))) \]

\[ [1 + A_\delta \exp(-\alpha_g(n - n_t)) \sin(2\pi(n - n_t)f)] , \]

(For \( n > n_t = 9600 \) \( G_\delta = 2, A_\delta = 0.5, f = 1/150, \alpha_\delta = 1/(15*150) \))
Emittances and rms beam sizes (No space charge)

No sextupole
Summaries of FNAL Booster measurements:

1. We are able to fit experimental measured IPM data (vertical and horizontal) to deduce the betatron emittances of the beam. We note that the vertical emittance increases rapidly in the initial stage of acceleration. The horizontal emittance seems to be less affected by the space charge at the initial stage.

2. The bunch width oscillation after the transition crossing has been used to deduce the bunch-mismatch factor, and the result is consistent with the rf program in the Booster operation.

3. We use the rms space charge force model to carry out numerical simulations. We are able to fit the observed data of emittance growth. The emittance growth in the vertical plane has resulted mainly from the skew quadrupoles, that induce a sum resonance at $Q_x+Q_z=$integer. This induces emittance growth mainly in the vertical plane. The reason is that the Montague resonance suppresses the growth of the horizontal emittance, and enhance the vertical emittance by about 25%. The random dipole field error also generates about 25% vertical emittance growth.
Systematic Resonances

- **Sp Ch potential:**
  \[ V_{sc}(x, z) = \frac{K_{sc}}{2} \int_0^\infty \frac{\exp \left[ -\frac{x^2}{2\sigma_x^2 + t} - \frac{z^2}{2\sigma_z^2 + t} \right] - 1}{\sqrt{(2\sigma_x^2 + t)(2\sigma_z^2 + t)}} \, dt, \quad K_{sc} = \frac{2Nr_0}{\beta^2 \gamma^3} \]

- **Expansion:** \((r = \sigma_z/\sigma_x)\)
  \[
  V_{sc}(x, z) = -\frac{K_{sc}}{2} \left\{ \left[ \frac{x^2}{\sigma_x(\sigma_x + \sigma_z)} + \frac{x^2}{\sigma_x(\sigma_x + \sigma_z)} \right] - \frac{1}{\sigma_z^2(\sigma_x + \sigma_z)^2} \left[ \frac{2 + r}{3} x^4 + \frac{2r}{r^3} x^2 z^2 + \frac{1 + 2r}{3r^3} z^4 \right] \right. \\
  + \left. \frac{1}{72\sigma_x^3(\sigma_x + \sigma_z)^3} \left[ \frac{8 + 9r + 3r^2}{5} x^6 + \frac{3(3 + r)r}{r^3} x^4 z^2 + \frac{3(3r + 1)}{r^3} x^2 z^4 + \frac{8r^2 + 9r + 3}{5r^5} z^6 \right] + \ldots \right\}
  \]

  \[
  V_{sc} = K_{sc} \int_r^1 \frac{1}{1 - v^2} \{ \exp[-a^2(1 - v^2) - b^2(\frac{1}{v^2} - 1)] - 1 \} dv
  \]

  \[
  a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, \quad b = \frac{z}{\sqrt{2(\sigma_x^2 - \sigma_z^2)}}, \quad v^2 = \frac{2\sigma_z^2 + t}{2\sigma_x^2 + t}
  \]
In action-angle variables,

\[ V_{sc,6}(J_x, J_z, \psi_x, \psi_z, \theta) \approx -\frac{1}{R} \sum_{\ell} |G_{60\ell}| J_x^3 \cos(6\psi_x - \ell \theta + \chi_{60\ell}) \]

\[ \quad - \frac{1}{R} \sum_{\ell} |G_{06\ell}| J_z^3 \cos(6\psi_z - \ell \theta + \chi_{06\ell}) - \cdots \]

\[ G_{60\ell} = \frac{1}{5760\pi} \int K_{sc} \beta_x^3 \left(8\sigma_x^2 + 9\sigma_x\sigma_z + 3\sigma_z^2\right) e^{i(6\phi_x - 6\nu_x\theta + \ell \theta)} ds \]

\[ G_{06\ell} = \frac{1}{5760\pi} \int K_{sc} \beta_z^3 \left(8\sigma_z^2 + 9\sigma_x\sigma_z + 3\sigma_x^2\right) e^{i(6\phi_z - 6\nu_z\theta + \ell \theta)} ds \]

Can be factored out sp-ch dependent part of resonance strength, giving dimensionless reduced strength \( g_{mnl} \):

\[ G_{mnl} = g_{mnl} \frac{K_{sc} R}{4\epsilon_{rms}^3} \]
Simulation Parameters

- Proton injection rate: particles per turn: \( 4 \times 10^{11} \)
- Harmonic number \( h = 84 \).
- Circumference \( C = 474.2 \text{m} \).
- Bunching factor = 2

\[
B = \frac{C}{(h\sqrt{2\pi}\sigma_s)} = 2
\]

- Initial emittance: \( \varepsilon_{N,rms} = 8.33 \times 10^{-6} \text{m} \)
- Kinetic energy is kept constant at 1GeV during the tracking.
Sample Simulation

\[ \Delta \nu_{sc,x} = 0.310 \quad \Delta \nu_{sc,z} = 0.290 \]

- 100 turns injection
- Systematic resonances:
  - \( 6n_x = P, \ 6n_z = P, \quad P = 24 \)
  - \( \frac{dn_{x,z}}{dn} = -0.0004 \)
- Aperture = 500 \( \pi \) mm-mrad
- Emittance growth factor (EGF):
  Final emit./initial emit.

\[ (v_{x0}, v_{z0}) = (4.25, 4.45) \]
\[ \Downarrow \]

\[ (v_x, v_z) = (3.69, 3.89) \]

\[ | g_{60P} | = 0.00181 \]
\[ | g_{06P} | = 0.00172 \]
Scaling Power Laws

- Tune ramp rate: $0.0004 \rightarrow 0.0112$
- Log-log plots
- Scaling Law
  \[ EGF = (-\frac{dv}{dn})^{-a} \]
Critical Tune Ramp Rate

\[ G_{06P/06P} \propto \Delta \nu_{sc,x/z} \]

The number of turns that the beam is under the influence of the resonance:

\[ \Delta n \approx \Delta \nu_{sc} / \left| \frac{d\nu}{dn} \right| \]

\[ \left| \frac{d\nu}{dn} \right|_c \approx 65 \ g_{60P} (\Delta \nu_{sc})^2 \]

Can serve as a guideline for FFAG design!
4th order Systematic Resonances

\[ \left| \frac{d
u}{dn} \right|_c \approx 8.4 \left( \Delta \nu_{sc} \right)^2 g_{04P} \exp\{31g_{04P}\} \]
Conclusion:

1. We carried out systematic measurements of the emittance evolution at the Fermilab Booster in order to model the effect of space charge force and other effects on beams. We found that the random skew quadrupoles were important on the emittance dilution. Stopband correction experiments would be very important to confirm our results. **Systematic measurements at other accelerators would be very valuable!**

2. Systematic Nonlinear space charge resonances can be important in high intensity accelerators. For future neutron source design, one should try to avoid the systematic nonlinear space charge resonance, if it is possible! Otherwise, minimize space charge resonance strengths!

3. For the non-scaling FFAG, the nonlinear resonances induced by the space charge potential can be the limiting factor. These resonances limit the phase advance of each basic cell to within $\pi/2$ to $\pi/3$, and thus the momentum acceptance is highly constrained.

4. We also find that the emittance growth factor for quadrupole and skew-quadrupole errors obeys a simple scaling law.

$$\text{EGF} = \exp \left[ \frac{\lambda 2 \pi g^2}{dn / d\nu} \right]$$

Many thanks for your attention
Comments after the workshop discussions

1. Using Quadrupole transfer function to measure the beam emittances


2. Using the measured betatron tune to determine the ring impedance

   \[ \omega = \omega_{n,w0} + j \frac{e\beta I Z_\perp}{4\pi R \gamma m Q \omega_0}. \]  
   \hspace{1cm} (2.318)

3. Once the barrier bucket is implemented, there is an opportunity to study head-tail effects of high intensity beams.